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Enumeration of Limit Cycles in Noncylindrical Cellular Automata

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Recently a method has been developed by Jen to enumerate limit cycles in cellular automata (CA) with periodic boundary conditions. This involves operations on a connectivity matrix whose elements are related to the invariance of a site in a particular neighborhood to application of the CA rule. We extend this method to the case of fixed boundary conditions, of interest in simulations. In this case, translational invariance is lost, and the enumeration procedure is much more tedious than with periodic boundary conditions. We show examples for a fixed-point, a period-two, and a period-three enumeration in considerable detail, and give results—in agreement with simulations—for the number of fixed points and period-two cycles in selected two-state, nearest-neighbor CA rules.

KEY WORDS: Elementary cellular automata; connectivity matrix; fixed boundary conditions.

1. INTRODUCTION

Cellular automata (CA)[1-6] are models discrete in space, time, and state variable, and therefore seem naturally suited for computer simulation. At the same time as efforts are being made to simulate and analyze physically meaningful models, there have been a few rigorous studies[7-9] of the long-time properties of simpler CA rules, especially of the determination of the number of limit cycles of a prescribed length.

In the present paper we extend the scope of these studies from the usual case of periodic boundary conditions to the case of fixed boundary conditions. The motivation is threefold: (1) Simulation in complex geometries with little or no increase in computation time has proved to be one of the strong points of CA; we should mention in particular simulations of fluid flow through porous media[10,11] which require fixed bound-

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aries. (2) Recent studies of CA with fixed boundaries have shown a wealth of previously unobserved and physically interesting behavior, such as global[12,13] and bistable[14] phase-space structures. (3) Fixed boundary conditions appear to play an important role in the critical behavior of some sandpile models.[15] The only studies of fixed boundary conditions in CA of which we are aware are an interesting but rather *ad hoc* analysis of CA rule 232 by Kerszberg and Mukamel[16] and a group-theoretic study of rule 150 with zero boundary conditions by Pitsianis and co-workers.[17] What we report here is a more general method, valid for arbitrary CA rules.

This paper will proceed as follows: in Section 2 we review Jen's[9] connectivity matrix formalism, which we extend to the case of fixed boundary conditions; we also present explicit calculations for a fixed-point, a period-two, and a period-three enumeration. In Section 3 we present a summary of results for short cycles in two-state nearest-neighbor rules, and we conclude with a discussion in Section 4. Appendices A and B provide details of the enumeration of period-3 cycles for rule 43, and of recursion relations for small cycles in elementary CA rules, respectively.

2. METHOD AND EXAMPLES

Consider a one-dimensional cellular automaton rule of radius r and k states per site, defined thus:

$$x_i^{t+1} = f(x_{i-r}^t, \ldots, x_i^t, \ldots, x_{i+r}^t); \qquad f\colon Z_k^{2r+1} \to Z_k \tag{1}$$

where $x_i^t \in Z_k$ is the value at the ith site at time t. The index i runs from 1 to L. The fixed boundary conditions are defined by the values of x_{1-r} to x_0 and x_{L+1} to x_{L+r}.

The connectivity matrix, as defined in ref. 9, is calculated as follows. Consider all k^{2r} possible states of $2r$ sites, and label each of these states thus:

$$i = \{i_1, i_2, \ldots, i_{2r}\}, \qquad i_m \in Z_k, \quad m = 1, \ldots, 2r \tag{2}$$

So, i labels the state as a whole, and the set of quantities $\{i_m\colon m = 1 \text{ to } 2r\}$ gives the values of sites that compose the state i. The connectivity matrix $A^{[1]}$ is defined thus:

$A_{ij}^{[1]} = 1$ iff

(a) $i_2 = j_1, i_3 = j_2, \ldots, i_{2r} = j_{2r-1}$ and

(b) $i_{r+1} = f(i_1, i_2, \ldots, i_{2r}, j_{2r})$

$A_{ij}^{[1]} = 0$ otherwise

So, a nonzero element of $A^{[1]}$ corresponds to a sequence of $(2r+1)$ sites, where the central site is invariant under the action of the rule. Taking powers of $A^{[1]}$ then extends this sequence in space, and, as shown by Jen,[9] the value of $(A^{[1]})^n_{ij}$ gives the number of sequences of length $(2r+n)$, beginning with i and ending with j, where the central n elements are unchanged under the action of the rule. The number of states invariant under one application of the rule (fixed points) for the case of periodic boundary conditions, and system size L, is the value of the trace of $(A^{[1]})^L$. However, we are interested in the case of fixed boundaries. In a similar manner, the number of fixed points for a one-dimensional cellular automaton with fixed boundaries and length L is given by

$$T_L^{[1]} = \sum_{i,j} (A^{[1]})^L_{ij} \tag{3}$$

where

$$i_{r-m+1} = x_{1-m} \qquad \text{and} \qquad j_{2r-1-m} = x_{L+m} \qquad \text{for} \quad m = 1 \text{ to } r$$

This is a rather more complicated expression than in the periodic case. The lack of translational invariance, plus the additional degrees of freedom allowed by the choice of boundary conditions, give us a general sum over elements of the matrix rather than a simple trace. However, in many cases, it is still possible to obtain an analytic result provided that k^{2r} is not too large. For the simplest case ($k=2$, $r=1$) we have solutions for the number of fixed points for all boundary conditions, all rules, and all lattice sizes. (The results for a selection of rules, for which we also have solutions for the numbers of period-two cycles, are given in Appendix B.)

To extend the method to cycles of arbitrary period, we have to consider the pth composition of the rule (which gives a compound rule of radius pr). It is clear that the evolution of all sites over p time steps can be described by such rules. However, the evolution of sites within a distance pr of either boundary will not be described by the same compound rule as that which describes the evolution of sites distant from the boundary, since some of the sites within the radius of the compound rule are, by virtue of being part of the boundary, fixed. This requires the introduction of additional, position-dependent, connectivity matrices.

Consider the evolution over p time steps of a system of width $L > 2(pr-1)$. For sites within a distance pr of the boundaries we label the compound rules thus:

$$x_i^{t+p} = l_i^{[p]}(x^t_{i-pr}, \ldots, x^t_i, \ldots, x^t_{i+pr}); \qquad i = 1, \ldots, (pr-1) \tag{4}$$

and

$$x^{t+p}_{L+1-i} = r_i^{[p]}(x^t_{L+1-i-pr}, \ldots, x^t_{L+1-r}, \ldots, x^t_{L+1-i+pr}); \qquad i = 1, \ldots, (pr-1) \tag{5}$$

Although the evolution does not depend on the boundary values $\{x_i: i < (1-pr) \text{ or } i > (L+pr)\}$, for ease of calculation we will include these sites and fix their values to be zero.

For the remaining sites, we have the bulk pth composition of the original rule:

$$x_i^{t+p} = a^{[p]}(x_{i-pr}^t, \ldots, x_i^t, \ldots, x_{i+pr}^t) \tag{6}$$

We now have $(2pr-1)$ rules of radius pr. For each rule we construct a connectivity matrix as before. ($a^{[p]} \to A^{[p]}$, $l_i^{[p]} \to L_i^{[p]}$ and $r_i^{[p]} \to R_i^{[p]}$.) We then define the following quantity:

$$T_L^{[p]} = \sum_{m,n} \left[\left(\prod_{i=1}^{pr-1} L_i^{[p]} \right) (A^{[p]})^{L-2pr+2} \left(\prod_{i=1}^{pr-1} R_{pr-i}^{[p]} \right) \right]_{mn} \tag{7}$$

This rather more complicated expression is a generalization of Eq. (3) to include the position-dependent connectivity matrices. It gives the number of sequences of length $(2pr+L)$ beginning with m and ending with n, where the central L elements are unchanged after applying the original rule p times. As before, the values of m and n must be taken to match the required boundary conditions. Although the values of $\{x_i: i < (1-pr) \text{ or } i > (L+pr)\}$ do not affect the evolution, they do restrict the allowed values of m and n and prevent overcounting.

As examples of the application of this technique, we will now calculate the number of fixed points and period-two limit cycles for the two-state nearest-neighbor rule 37 in Wolfram's nomenclature,[3] and the number of period-three cycles for rule 43.

2.1. Rule 37: Fixed Points

This rule is defined as follows:

$$\begin{aligned} f(101, 010, 000) &= 1 \\ f(001, 100, 011, 110, 111) &= 0 \end{aligned} \tag{8}$$

Note that the rule is invariant under the interchange of left and right. From Eq. (2) we label the possible values of central and peripheral states as follows:

$$\begin{aligned} i=0: &\quad i_1=0, \quad i_2=0 \\ i=1: &\quad i_1=0, \quad i_2=1 \\ i=2: &\quad i_1=1, \quad i_2=0 \\ i=3: &\quad i_1=1, \quad i_2=1 \end{aligned} \tag{9}$$

Directly from the definition, the nonzero elements of $A^{[1]}$ are

$$(i, j) = (0, 1), \quad (1, 2), \quad (2, 0) \tag{10}$$

Inserting the explicit form of $A^{[1]}$ into the equation

$$(A^{[1]})^{n+1} = A^{[1]}(A^{[1]})^n \tag{11}$$

yields the following recursion relations:

$$\begin{aligned} A_0^{n+1} &= A_1^n \\ A_1^{n+1} &= A_2^n \\ A_2^{n+1} &= A_0^n \end{aligned} \tag{12}$$

where A_i^n denotes the ith row of $(A^{[1]})^n$. The solution for an arbitrary power of $A^{[1]}$ can then be written thus:

$$\begin{aligned} A_0^{3n} &= A_2^{3n+1} = A_1^{3n+2} = (1, 0, 0, 0) \\ A_1^{3n} &= A_0^{3n+1} = A_2^{3n+2} = (0, 1, 0, 0) \\ A_2^{3n} &= A_1^{3n+1} = A_0^{3n+2} = (0, 0, 1, 0) \end{aligned} \tag{13}$$

where $(\cdots)$ denotes the elements of a row. The numbers of fixed points for given fixed boundary conditions is then calculated by taking the appropriate sum over elements as detailed in Eq. (3). The results for all fixed boundary conditions (plus the periodic case for comparison) and all values of L are given in Table I.

Note that the results for (01) and (10) boundary conditions are identical, as they must be for a left–right symmetric rule. These values agree with computational results[12] which have been performed up to $L = 14$.

Table I. Numbers of Fixed Points for Rule 37, As a Function of Lattice Size L, for All Fixed Plus Periodic Boundary Conditions

x_0	x_{L+1}	$L = 3n$	$L = 3n+1$	$L = 3n+2$
0	0	1	1	2
0	1	1	1	0
1	0	1	1	0
1	1	0	0	1
Periodic		3	0	0

2.2. Rule 37: Period two Limit Cycles

$L^{[2]}$ and $R^{[2]}$ are calculated by considering all possible sets of values for five sites. All elements corresponding to $x_{-1}=1$ or $x_{L+2}=1$ are set to zero to avoid overcounting. Since the rule is symmetric, the non-zero elements of $R^{[2]}$ follow directly from those of $L^{[2]}$. Thus:

$$\begin{aligned} f^2(\hat{0}\hat{0}000, \hat{0}\hat{0}010, \hat{0}\hat{0}101) &= 0 \\ f^2(\hat{0}\hat{0}001, \hat{0}\hat{0}100, \hat{0}\hat{0}011, \hat{0}\hat{0}110, \hat{0}\hat{0}111) &= 1 \\ f^2(\hat{0}\hat{1}001, \hat{0}\hat{1}100, \hat{0}\hat{1}011, \hat{0}\hat{1}110, \hat{0}\hat{1}010, \hat{0}\hat{1}111) &= 0 \\ f^2(\hat{0}\hat{1}000, \hat{0}\hat{1}101) &= 1 \end{aligned} \tag{14}$$

where $\hat{0}$ or $\hat{1}$ indicates a boundary (constant) bit. We label the possible values of four bits thus:

$$\begin{aligned} i=0: &\quad 0000 \\ i=1: &\quad 0001 \\ &\vdots \\ &\vdots \\ i=15: &\quad 1111 \end{aligned} \tag{15}$$

By direct application of the rule, the nonzero elements of $L^{[2]}$ and $R^{[2]}$ are:

$$\begin{aligned} &L_{ij}^{[2]}: \\ &(i,j) = (0,0), (1,2), (2,4), (3,6), \\ &\qquad (3,7), (4,9), (5,10), (5,11) \text{ and } (6,13) \\ &R_{ij}^{[2]}: \\ &(i,j) = (0,0), (2,4), (4,8), (5,10), \\ &\qquad (6,12), (9,2), (11,6), (13,10), (14,12) \end{aligned} \tag{16}$$

The expression for the number of invariant states then reduces to

$$T_L^{[p]} = \sum_{i,j} (A^{[2]})_{ij}^{L-2} \tag{17}$$

where:

If $x_0 = 0$, $i = 0$, 2, 4, 6, or 7.
If $x_0 = 1$, $i = 9$, 10, 11, or 13.
If $x_{L+1} = 0$, $j = 0$, 2, 4, 6, or 14.
If $x_{L+1} = 1$, $j = 5$, 9, 11, or 13.

By applying the rule to all possible initial states composed of five bits, the nonzero elements of $A^{[2]}$ are

$$\begin{aligned}(i, j) = {} & (0, 0), (0, 1), (1, 3), (2, 4),\\ & (3, 6), (3, 7), (4, 9), (5, 10),\\ & (5, 11), (6, 12), (7, 14), (7, 15),\\ & (8, 0), (9, 2), (12, 8), (13, 10),\\ & (14, 12), (15, 14), (15, 15)\end{aligned}$$

If we denote the ith row of $(A^{[2]})^n$ by A_i^n, then for $n \geqslant 6$, we have the following set of recursion relations:

$$\begin{array}{lll}
A_0^n = A_0^{n-1} + A_1^{n-1} & A_5^n = 0 & A_{10}^n = A_{11}^n \\
A_1^n = A_1^{n-1} + A_0^{n-5} & A_6^n = A_0^{n-3} & A_{12}^n = A_0^{n-2} \\
A_2^n = A_4^{n-1} & A_7^n = A_1^{n+1} & A_{13}^n = 0 \\
A_3^n = A_1^{n+1} & A_8^n = A_0^{n-1} & A_{14}^n \equiv A_6^n \\
A_4^n = A_4^{n-3} & A_9^n = A_4^{n-2} & A_{15}^n \equiv A_7^n
\end{array}$$

This yields the solution:

$$\begin{aligned}
A_4^{3n} &= (0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)\\
A_4^{3n+1} &= (0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0)\\
A_4^{3n+2} &= (0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0) \qquad (18)\\
A_0^n &= (z_{n+3}, z_{n+2}, 0, z_{n+1}, 0, 0, z_n, z_n, y_{n+2}, 0, 0, 0, y_{n+3}, 0, y_{n+3}, y_{n+3})\\
A_1^n &= (y_{n+2}, y_{n+1}, 0, y_n, 0, 0, y_{n-1}, y_{n-1}, z_{n-1}, 0, 0, 0, z_n, 0, z_n, z_n)
\end{aligned}$$

where

$$\begin{aligned} z_n &= z_{n-1} + y_{n-2}\\ y_n &= y_{n-1} + z_{n-4}\end{aligned} \qquad (19)$$

The initial values of z_n and y_n are calculated by performing the multiplication explicitly for the lowest powers of the matrix. The results are given in Table II.

The number of states which lie in period-two limit cycles is obtained by subtracting the number of fixed points (states which are clearly also invariant under two applications of the rule). This gives the following

Table II. The Initial Values of z_n and y_n as Defined in Equation 18

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14
z_n	0	0	1	1	1	1	1	1	2	4	7	11	16	22
y_n	1	0	0	0	0	0	1	2	3	4	5	6	8	12

solution for the number of period-two cycles as a function of L, for $x_0 = 0$, $x_{L+1} = 0$:

$$z_{L-2} + y_{L-2} + y_{L+1} + \tfrac{1}{2}[z_{L-5} + z_{L-1} + z_{L+1}] \tag{20}$$

and also for the periodic case:

$$z_{L+3} + 2y_{L+1} \tag{21}$$

both of which agree with computational results. There are no period-two cycles for the other possible fixed boundary conditions.

2.3. Rule 43: Period-Three Limit Cycles

Rule 43 is defined thus:

$$\begin{aligned} f(101, 011, 001, 000) &= 1 \\ f(010, 100, 110, 111) &= 0 \end{aligned} \tag{22}$$

and is hence invariant under conjugation (all 1's change to 0's and *vice versa*). This means that we need only calculate the results for $x_{L+1} = 0$ and $x_0 = 0$ or 1. For the case of fixed points and period-two limit cycles, the calculation proceeds as before. We find that in the case of fixed boundaries, for $L \geqslant 4$, there are no fixed points or period-two limit cycles. In the periodic case, there are no fixed points for any L, and either one or two period-two limit cycles, for odd or even values of L, respectively. We now proceed to the enumeration of the period-three limit cycles. Since this involves five 64×64 matrices, we will not give all the details here (see Appendix A).

From Eq. (7) the number of states invariant under three iterations of the rule can be written thus:

$$T_L^{[3]} = \sum_{i,j} [(A^{[3]})^{L-4}]_{ij} \tag{23}$$

where the suffixes label all sets of possible values for six sites:

$$
\begin{array}{ll}
i=0: & 000000 \\
i=1: & 000001 \\
\vdots & \\
\vdots & \\
i=64: & 111111
\end{array}
\tag{24}
$$

and the allowed values of i and j are deduced from the matrices $L_1^{[3]}$, $L_2^{[3]}$, $R_1^{[3]}$, and $R_2^{[3]}$ as defined in Eqs. (4) and (5).

The matrix $A^{[3]}$ has 80 nonzero elements. Solving for $(A^{[3]})^n$ row-by-row, and using A_i^n to denote the ith row, we have the following key recursion relations:

$$
\begin{aligned}
A_4^n &= A_4^{n-3} \\
A_{13}^n &= A_{13}^{n-3} \\
A_7^n &= A_{13}^n + A_{13}^{n-1} + A_{13}^{n-2} \\
&\quad + A_4^n + A_4^{n-1} + A_4^{n-2} + A_7^{n-6} \\
&\equiv C + A_7^{n-6}
\end{aligned}
\tag{25}
$$

Table III. The Numbers of Period Three Cycles for Rule 43 as a Function of System Size *L*

x_0	x_{L+1}	L
0	0	$\frac{L}{2}$ or $\frac{L-1}{2}$
0	1	$\frac{L}{2}$ or $\frac{L+1}{2}$
1	0	$\frac{L}{2}$ or $\frac{L+1}{2}$
1	1	$\frac{L}{2}$ or $\frac{L-1}{2}$
Periodic		If $L=6m:4$ If $L=3(2m+1):2$

where $n \geqslant 10$ and C denotes a constant row. At sufficiently large n, all other rows can be expressed in terms of these three. The results for the numbers of period-three cycles, for all boundary conditions, as a function of the system size L are given in Table III.

3. SUMMARY OF RESULTS

In Appendix B we tabulate the results obtained for the numbers of period-one and period-two cycles for a selection of rules. We include the results for all fixed boundaries, plus the periodic case for comparison. We see that in many cases, the result in the periodic case can be expressed in terms of the same quantities used to express the results for fixed boundaries, and is not bounded in any way by the exact values in the fixed case. This is not surprising, since the series obtained to express these results have their origins in the recursion relations which describe the relations between the rows of the connectivity matrix raised to various powers. We also note here that all the results for a given rule will obey the recursion relation which can be derived from the characteristic equation of $A^{[p]}$.[9] For the case of fixed points, for $r = 1$, this is a feasible approach: the characteristic equation will yield a recursion relation which contains at most four terms. However, for the case of period-two cycles, the same approach will yield a relation containing up to 15 terms. For the rules listed, we found that the relations obtained contained fewer terms than this, and could be extracted from the elements of $A^{[2]}$ with only moderate effort. Although the rules dealt with in the main text both possess symmetries, this is not a necessary requirement. As the period of the required cycles increases, the algebraic manipulation required to obtain *analytic* results becomes increasingly cumbersome due to the size of the matrices. However, it is still possible to obtain *numerical* results on a computer.

Negative Results

It is clear from the results shown in Appendix B that fixed boundaries can affect greatly the long-time properties of a CA rule. Since the connectivity matrix is unchanged by the boundary conditions, the difference with periodic boundary conditions must arise either from the left–right boundary connectivity matrices (L and R in this paper) or from summing over specific matrix elements [see Eq. (7)] other than the trace.

Elementary CA rules provide a dramatic example of this: Jen has proved[18] that with periodic boundary conditions *any* rule must have at least one fixed point or period-two cycle for *any* lattice size. Her proof can be recast in terms of the properties of the trace of the connectivity matrix.

With fixed boundary conditions there are a number of combinations of rules and boundaries which, for large enough lattice size, have *neither* fixed points nor period-two cycles.

These rules are as follows:

Rule:	$(x_0 x_{L+1})$
2:	(01), (11)
10:	(01), (11)
11:	(00), (01), (10), (11)
24:	(10), (11)
25:	(00), (01)
26:	(01), (11)
27:	(00), (01)
33:	(01), (10)
35:	(10)
38:	(01), (11)
41:	(01), (10), (11)
42:	(01), (11)
43:	(00), (01), (10), (11)
46:	(01), (11)
54:	(01), (10)
56:	(11)
57:	(00), (01), (11)
58:	(01), (11)
60:	(10), (11)
62:	(01), (10), (11)
106:	(01)
122:	(01), (10)
130:	(01)
152:	(10)

We choose as a first example rules 2, 10, 38, and 46, since the powers of the connectivity matrix $(A^{[2]})^n$ are identical for $n > 4$. The nonzero elements of this matrix are $(i, j) = (0, 0)$, $(4, 0)$, $(8. 0)$, $(12, 0) = 1$ for any $n > 4$. In the periodic boundary case, the number of states belonging in cycles no longer than two is $\mathrm{Tr}(A^{[2]})^n = 1$. For the case of fixed boundaries with $x_{L+1} = 1$, Eq. (7) reduces to a sum over elements $(A^{[2]})^n_{ij}$, with j taking only odd values. Independently of the R and L matrices, the sum over odd j readily gives *no* fixed points or period-two cycles for lattice sizes $n > 4$, in agreement with simulations.

Similarly, rule 11 has the following nonzero elements in the third or higher powers of the $A^{[2]}$ connectivity matrix:

$$(i, j) = (0, 0), (0, 1), (0, 2), (0, 3),$$
$$(15, 12), (15, 13), (15, 14), (15, 15)$$

Since the L matrix only has nonzero elements in columns 2, 4, 5, 8, 9, 10, and 11, Eq. (7) yields *no* fixed points or period-two limit cycles for *any* combination of fixed boundaries.

For the remaining rules which do not appear in Appendix B, similar arguments apply. For rules 27, 60, 62, and 130, $(A^{[2]})^n$ evolves to a fixed form. For rules 42, 56, 57, and 58, it alternates between two fixed matrices, depending on the value of n modulo 2. For rule 152, it evolves to a fixed form, except for a few terms which are of order n. Rule 35 leads to a matrix with fixed, order n and alternating terms. For the last rule, 106, the matrix takes on one of four fixed values, depending on the value of n modulo 4. Despite this variation in behavior, it is simple to verify the absence of both fixed points and period-two cycles for all the above rules: neither the changing elements nor the nonzero constant elements contribute to the relevant sums.

4. DISCUSSION

In the present paper we have extended the connectivity matrix technique for enumerating limit cycles in cellular automata (CA) to include the case of fixed boundary conditions, which is what is often done in simulations of physical interest. Jen's original technique[9] only dealt explicitly with limit cycles in cylindrical CA, i.e., one-dimensional with periodic boundary conditions.

With fixed boundary conditions the translational invariance of the system is lost. It is then necessary (1) to introduce additional, position-dependent matrices for the sites nearest to the boundary for limit cycles longer than one, and (2) to sum over particular elements of the powers of the connectivity matrix rather than to just take traces. We have shown explicit examples of how the recursion relations are obtained for limit cycles of length one, two, and three in Section 2. The calculation of these is much more tedious than in the cylindrical case.

We have also included in Section 3 a summary of results for small cycles (length one or two) for a few interesting elementary CA (two states, nearest neighbor) rules. As a striking example of the effects of translational invariance, we provide a list of rules which exhibit *no* cycles of length less than three for large enough lattices; this cannot occur with periodic boundary conditions.

APPENDIX A

We give here the details of the analysis of period-three cycles for rule 43. As in Eq. (23), the number of states invariant under three iterations of the rule is

$$T_L^{[3]} = \sum_{i,j} [(A^{[3]})^{L-4}]_{ij}$$

where the indices take the following values:

If $x_0 = 0$, then $i = 0$, 1, 2, 3, 4, 6, 7, 13, 16, 17, 18, 19, 22, 23, 24, 25, 27, 28, 29, 30, or 31.

If $x_0 = 1$, then $i = 32$, 33, 34, 35, 36, 38, 39, 40, 41, 44, 45, 46, 47, 50, 56, 57, 59, 60, 61, 62, or 63.

If $x_{L+1} = 0$, then $j = 4$, 6, 8, 10, 12, 16, 18, 20, 24, 26, 28, 34, 36, 40, 42, 44, 48, 50, 52, 56, or 58.

All elements of $A^{[3]}$ are zero, except for those whose row and column are given in Table IV, and which take the value one. This leads to the following set of relations for individual rows of $(A^{[3]})^n$, for n sufficiently large, using the same notation as in the text:

$$A_4^n = A_4^{n-3} \qquad A_5^n = A_{13}^n \qquad A_6^n = A_4^{n-1} + A_{13}^{n-1} \qquad A_7^n = C + A_7^{n-6}$$

$$A_9^n = A_4^{n+1} \qquad A_{10}^n = A_4^{n-1} \qquad A_{11}^n = A_{13}^{n+1} \qquad A_{12}^n = A_4^n$$

$$A_{13}^n = A_{13}^{n-3} \qquad A_{17}^n = A_4^{n-2} + A_4^{n-3} + A_{13}^{n-3} + A_7^{n-2} \qquad A_{18}^n = A_4^{n+2}$$

Table IV.

(4, 8)	(4, 9)	(5, 11)	(6, 12)	(6, 13)	(7, 14)	(7, 15)	(9, 18)
(10, 20)	(11, 22)	(11, 23)	(12, 24)	(12, 25)	(13, 27)	(14, 28)	(14, 29)
(15, 30)	(15, 31)	(16, 32)	(16, 33)	(17, 34)	(17, 35)	(18, 36)	(19, 38)
(19, 39)	(20, 40)	(20, 41)	(21, 43)	(22, 45)	(25, 50)	(26, 52)	(27, 54)
(27, 55)	(28, 56)	(28, 57)	(29, 59)	(30, 60)	(30, 61)	(31, 62)	(31, 63)
(32, 0)	(32, 1)	(33, 2)	(33, 3)	(34, 4)	(35, 6)	(35, 7)	(36, 8)
(36, 9)	(37, 11)	(38, 13)	(41, 18)	(42, 20)	(43, 22)	(43, 23)	(44, 24)
(44, 25)	(45, 27)	(46, 28)	(46, 29)	(47, 30)	(47, 31)	(48, 32)	(48, 33)
(49, 34)	(49, 35)	(50, 36)	(51, 38)	(51, 39)	(52, 40)	(52, 41)	(53, 43)
(54, 45)	(56, 48)	(56, 49)	(57, 50)	(57, 51)	(58, 52)	(59, 54)	(59, 55)

$A^n_{19} = A^{n-2}_{13}$ $A^n_{20} = A^n_4$ $A^n_{21} = A^n_{13}$ $A^n_{22} = A^{n+2}_{13}$

$A^n_{25} = A^{n+1}_4$ $A^n_{26} = A^{n-1}_4$ $A^n_{27} = A^{n+1}_{13}$ $A^n_{28} = A^{n+2}_7 - A^n_{13}$

$A^n_{29} = A^n_{13}$ $A^n_{34} = A^{n-1}_4$ $A^n_{35} = A^{n-2}_4 + A^{n-2}_{13} + A^{n-1}_7$

$A^n_{36} \equiv A^n_4$ $A^n_{37} \equiv A^n_5$ $A^n_{38} = A^{n-1}_{13}$ $A^n_{41} \equiv A^n_9$

$A^n_{42} \equiv A^n_{10}$ $A^n_{43} \equiv A^n_{11}$ $A^n_{44} \equiv A^n_{12}$ $A^n_{45} \equiv A^n_{13}$

$A^n_{46} \equiv A^n_{14}$ $A^n_{47} \equiv A^n_{15}$ $A^n_{48} \equiv A^n_{16}$ $A^n_{49} \equiv A^n_{17}$

$A^n_{50} \equiv A^n_{18}$ $A^n_{51} \equiv A^n_{19}$ $A^n_{52} \equiv A^n_{20}$ $A^n_{53} \equiv A^n_{21}$

$A^n_{54} \equiv A^n_{22}$ $A^n_{56} = A^{n-1}_{17}$ $A^n_{57} = A^{n-2}_4 + A^n_{13}$ $A^n_{58} \equiv A^n_{26}$

$A^n_{59} \equiv A^n_{27}$

Rows which do not appear in the above either zero initially, or evolve to zero after a few multiplications. All that remains is to detail the required initial conditions thus:

$$
\begin{aligned}
A^1_4 &= [8, 9] \\
A^2_4 &= [18] \\
A^3_4 &= [36] \\
A^1_{13} &= [27] \\
A^2_{13} &= [54, 55] \\
A^3_{13} &= [45] \\
C &= [8, 9, 18, 27, 36, 45, 54, 55] \\
A^1_7 &= [14, 15] \\
A^2_7 &= [28, 29, 30, 31] \\
A^3_7 &= [56, 57, 59, 60, 61, 62, 63] \\
A^4_7 &= [48, 49, 50, 51, 54, 55] \\
A^5_7 &= [32, 33, 34, 35, 36, 38, 39, 45] \\
A^6_7 &= [0, 1, 2, 3, 4, 6, 7, 8, 9, 13, 27] \\
A^7_7 &= [8, 9, 12, 13, 14, 15, 18, 27, 54, 55] \\
A^8_7 &= [18, 24, 25, 27, 28, 29, 30, 31, 36, 45, 54, 55] \\
A^9_7 &= [8, 9, 27, 36, 45, 50, 54, 55, 56, 57, 59, 60, 61, 62, 63]
\end{aligned}
\tag{26}
$$

Table V.

(x_0x_{L+1})	Fixed points	Period-2 cycles	Comments
#6			
(00)	$\frac{1}{2}[L+3-f_L]$	0	$f_L=\frac{1}{2}[1+(-1)^L]$
(01)	f_L	$1-f_L$	and for the rest
(10)	$\frac{1}{2}[L+1+f_L]$	0	of the table
(11)	$1-f_L$	f_L	
Periodic	$1+2f_L$	0	
#7			
(00)	1	$\frac{1}{2}[L-1+f_L]$	—
(01)	f_L	$\frac{1}{2}[L+1-f_L]$	
(10)	1	$\frac{1}{2}[L-1-f_L]$	
(11)	$1-f_L$	$\frac{1}{2}[L-1+f_L]$	
Periodic	$2f_L$	1	
#18			
(00)	1	a_{L-2}	$a_n=a_{n-2}+a_{n-3}$
(01)	0	a_{L-1}	$a_1=0$, $a_2=1$, $a_3=1$
(10)	0	a_{L-1}	
(11)	0	a_L	
Periodic	1	b_L	$b_n=b_{n-4}+2b_{n-5}+b_{n-6}$ $b_1,\ldots, b_6=0, 0, 0, 2, 5, 3$
#23			
(00)	$1-f_L$	c_{L-2}	$c_n=c_{n-1}+c_{n-2}$
(01)	f_L	c_{L-2}	$c_1=1$, $c_2=2$
(10)	f_L	c_{L-2}	
(11)	$1-f_L$	c_{L-2}	
Periodic	$2f_L$	d_L	$d_n=2d_{n-1}-d_{n-2}+d_{n-4}$ $d_1=1$, $d_2=1$, $d_3=1$, $d_4=3$
#29			
(00)	1	$\frac{1}{2}[e_{L+2}+e_L-1]$	$e_n\equiv g_n+h_n$
(01)	2	$e_{L+1}-1$	$g_n=g_{n-1}+g_{n-3}+h_{n-3}$
(10)	f_L	$\frac{1}{2}[e_{L+2}+e_{L+1}-f_L]$	$h_n=h_{n-1}+g_{n-2}+h_{n-2}$
(11)	1	$\frac{1}{2}[e_{L+2}+e_{L+1}-1]$	$g_2=1$, $g_3=1$, $g_4=1$
Periodic	$2f_L$	$g_{L+2}+g_L+h_{L+1}-f_L$	$h_2=0$, $h_3=0$
#30			
(00)	$2-f_L$	f_L	—
(01)	1	$1-f_L$	
(10)	f_L	$1-f_L$	
(11)	1	f_L	
Periodic	$1+2f_L$	0	
#33			
(00)	0	$\frac{1}{2}[k_{L+3}-k_{L+2}+k_{L+1}$ $k_L-k_{L-1}+k_{L-2}]$	$k_n=2k_{n-1}-k_{n-2}+k_{n-4}$ $k_1=k_2=k_3=k_4=1$
(01)	0	0	
(10)	0	0	
(11)	0	1	
Periodic	0	$2k_{L+1}-k_L+f_L$	

Table V. *(Continued)*

$(x_0 x_{L+1})$	Fixed points	Period-2 cycles	Comments
#41			
(00)	0	$1-f_L$	
(01)	0	0	
(10)	0	0	
(11)	0	0	
Periodic	0	$4\delta_{0r}+\delta_{1r}+2\delta_{2r}+\delta_{3r}$	$L=4m+r$ $r=0, 1, 2,$ or 3
#45			
(00)	$1+\delta_{2r}$	0	$L=3m+r$
(01)	$1+\delta_{1r}$	$1-\delta_{1r}$	$r=0, 1,$ or 2
(10)	$1-\delta_{2r}$	0	
(11)	$1-\delta_{1r}$	δ_{1r}	
Periodic	$3\delta_{0r}$	1	
#54			
(00)	1	0	
(01)	0	0	
(10)	0	0	
(11)	0	2	
Periodic	1	$2\delta_{0r}$	$L=4m+r$ $r=0, 1, 2,$ or 3
#110			
(00)	3	$\frac{1}{2}[L-1-f_L]$	
(01)	0	f_L	
(10)	2	$\frac{1}{2}[L-1+f_L]$	
(11)	0	$1-f_L$	
Periodic	1	$2\delta_{0r}$	$L=4m+r$ $r=0, 1, 2,$ or 3
#122			
(00)	1	1	
(01)	0	0	
(10)	0	0	
(11)	0	$m_{L+1}+m_{L+2}+m_{L+4}$	$m_n=m_{n-4}+2m_{n-5}+m_{n-6}$
Periodic	1	$f_L+2m_{L+3}+m_{L-2}+m_{L-3}$	$m_1,\dots, m_6=0, 0, 1, 0, 0, 0$
#126			
(00)	1	$p_{L+2}+p_L+p_{L-1}$	$p_n=p_{n-4}+2p_{n-5}+p_{n-6}$
(01)	0	$p_{L+3}+p_{L+1}+p_L$	$p_1,\dots, p_6=0, 0, 0, 0, 1, 1$
(10)	0	$p_{L+3}+p_{L+1}+p_L$	
(11)	0	$p_{L+4}+p_{L+2}+p_{L+1}$	
Periodic	1	$2p_{L+1}+3p_L$	
#146			
(00)	1	$p_{L+1}+p_{L-1}+p_{L-2}$	—
(01)	0	$p_{L-1}+p_{L-3}+p_{L-4}$	
(10)	0	$p_{L-1}+p_{L-3}+p_{L-4}$	
(11)	1	$p_{L-3}+p_{L-5}+p_{L-6}$	
Periodic	2	$2p_{L+1}+3p_L$	

where $[\cdots]$ denotes a row of 64 elements, with the only nonzero elements (1's) in the positions indicated. (Elements are labeled from 0 to 63.) This now gives a complete solution for $(A^{[31]})^n$ for all $n \geq 10$. For $n \leq 9$, we can perform the multiplication explicitly.

APPENDIX B

Table V contains the results for the numbers of fixed points and period-two cycles for a selection of elementary CA rules. Note that the numbers of period-two cycles do *not* include the fixed points, and that results do not necessarily apply to the smallest lattice sizes.

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